NaSTech
Journal of Natural Sciences and Technologies
2022, 1(1), 124-134., DOI: 10.5281/zenodo.7323506

ELECTROMAGNETIC ENERGY CONSERVATION BY BIQUATERNIONS

Nur ULUHAN[1], Abidin KILIÇ[2]

[1] Physics, Graduate Education Institute, Eskisehir Technical University, Eskisehir, TURKIYE
[2] Physics,Department, Eskisehir Technical University, Eskisehir, TURKIYE

ABSTRACT

In this document, after defining biquaternions algebra, Poynting Theorem is derived in this algebra. Because of 8-component biquaternions containing 3-dimensional vector space and 4-dimensional quaternion space, we can examine many physical quantities in biquaternion algebra. Based on this information, the generalized field Maxwell Equations and Gauge Transformations is showed in non-comutative but associative biquaternion algebra in homogenous media. Then, Noether and Poynting Theorems are introduced in terms of biquaternionic differential operator equation and used for deriving equations in electromagnetic energy conservation. In conclusion, it is seen that these biquaternionic equations can be derived from generalized 3-dimensional vector space in literature before.

Keywords: Quaternions, Biquaternions, Maxwell Equations, Poynting Theorem

INTRODUCTION

With the discovery of quaternions by William Hamilton in 1843, studies on complex numbers began to increase. Quaternions are a hyper-complex number system for studying 4-dimensional spacetime. **[1]** In the years when the quaternion algebra was discovered, British mathematicians such as A. Carley, K. Clifford and J. J. Slyvester used quaternions as well as physicists such as J. C. Maxwell and P. G. Tait, who built the electromagnetic theory continued their studies on electromagnetic issues by using quaternions. **[2]** Physicists adopted the use of vector and tensor algebra in the early 20th century. Until the middle of the 20th century, the practical use of quaternions was minimal compared to other methods. However, this situation has changed rapidly as progress has been made in these areas with the increasing use of quaternions in robotics, animation and computer technology. **[3]**

On the other hand biquaternions are the 8-dimensional form of quaternions. Its applications in physics are also taking place at an increasing rate in recent years. Gürlebeck and Wolfgang applied biquaternions to special theory of relativity, particle mechanics and electromagnetism and showed that biquaternions have a wide range of applications. **[4]**

Poynting's theorem includes an expression of energy density representing the dissipation of energy for time-varying electric and magnetic fields. It also says that the reduction in energy stored in the fields due to the work done on the charges by the electromagnetic force, minus the energy flowing out from the surface. Poynting's theorem is also identified in the literature as the work-energy theorem. **[5]**

In this paper, the history and algebra of quaternions and biquaternions are included. Then, Maxwell's equations, which are also valid for biquaternions, are obtained, and Poynting's theorem is defined for biquaternions.

QUATERNION ALGEBRA

A quaternion $\boldsymbol{\alpha}$ is defined as a complex number:

$$\boldsymbol{\alpha} = a_0 + a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k} \tag{2.1}$$

formed from 4 different units (1, ***i, j, k)*** by means of the real parameters a_i (i =0, 1, 2, 3), where ***i, j, k*** are 3-orthogonal unit elements. With ideas from both vector and matrix algebra, the quaternion $\boldsymbol{\alpha}$ may be viewed as a linear combination of scalar and base element a_0 and a vector $\vec{a}$:

$$\boldsymbol{\alpha} = a_0 + \vec{a}. \tag{2.2}$$

If $a_0 = 0$ is a real number and is called a *scalar quaternion*; when $\vec{a} = \vec{0}$ is a purely imaginary number and is called a *vector quaternion*. As we can observe, quaternions includes scalars and unit elements, and they are in the subspace of quaternions.

The conjugate of a quaternion $\boldsymbol{\alpha}$, denoted by $\boldsymbol{\alpha}^*$, is defined by negating its vector part (or imaginary part); that is

$$\boldsymbol{\alpha}^* = a_0 - \vec{a}. \tag{2.3}$$

It is convenient to represent, quaternions and their algebra in matrix form to simplfy equation manipulations. The matrix (column vector) representation of an arbitrary quaternion $\boldsymbol{\alpha}$ with respect to the basis (1, ***i, j, k)*** is merely the collection of its parameters:

$$\boldsymbol{\alpha} = [a_0, a_1, a_2, a_3]^T = [a_0, \boldsymbol{\alpha}^T]^T \tag{2.4}$$

where superscript T indicates the transpose of a matrix.

Since scalars and spatial vectors are in the subspace of quaternions, the rules in scalar and vector algebra alsa apply to quaternions. Let us consider the following three quaternions:

$\boldsymbol{\alpha} = a_0 + \vec{a}$, $\boldsymbol{\beta} = b_0 + \vec{b}$, $\boldsymbol{\gamma} = c_0 + \vec{c}$.

Addition and subtraction, ±, of two quaternions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are defined as

$$\boldsymbol{\alpha} \pm \boldsymbol{\beta} = (a_0 + b_0) + (\vec{a} \pm \vec{b}). \tag{2.5}$$

The quaternion addition and subtraction obey associative and commutative laws.

Quaternion multiplication, designated by ×, is defined as

$$\begin{aligned}\boldsymbol{\alpha} \times \boldsymbol{\beta} &= (a_0 + \vec{a}) \times (b_0 + \vec{b}) \\ &= a_0 \times b_0 + a_0 \times \vec{b} + b_0 \times \vec{a} + \vec{a} \times \vec{b}\end{aligned} \tag{2.6}$$

Where the scalar-scalar and scalar-vector quaternion products are defined, respectively, the same way as scalars and spatial vectors; thus $a_0 \times\times b_0 = a_0 b_0$, $a_0 \times \vec{b} = a_0\vec{b}$. The vector-vector quaternion product is defined as:

$$\vec{a} \times \vec{b} = - \vec{a} \,.\, \vec{b} + \vec{a} \times \vec{b} \tag{2.7}$$

Journal of Natural Sciences and Technologies
2022, 1(1), 124-134., DOI: 10.5281/zenodo.7323506

Where the operations "." and "×" define the dot product and the cross product in the space of unit elements.

The norm of a quaternion $\boldsymbol{\alpha}$, denoted by N($\boldsymbol{\alpha}$), is a scalar quaternion and is defined as N($\boldsymbol{\alpha}$) = $\boldsymbol{\alpha}^* \times \boldsymbol{\alpha} = \boldsymbol{\alpha} \times \boldsymbol{\alpha}^* = a_0^2 + a_1^2 + a_2^2 + a_3^2$.

Unlike unit elements, the set of quaternions form a division algebra, since for each non-zero quaternion $\boldsymbol{\alpha}$ there is an inverse $\boldsymbol{\alpha}^{-1}$ such that $\boldsymbol{\alpha} \times \boldsymbol{\alpha}^{-1} = \boldsymbol{\alpha}^{-1} \times \boldsymbol{\alpha} = 1$. Consider two non-zero quaternions $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} = \frac{\boldsymbol{\alpha}^*}{N\,(\boldsymbol{\alpha})}$ Since $\boldsymbol{\alpha} \times \boldsymbol{\beta} = \frac{\boldsymbol{\alpha} \times \boldsymbol{\alpha}^*}{N\,(\boldsymbol{\alpha})} = 1$, we find the inverse of $\boldsymbol{\alpha}$ to be **[6]**

$$\boldsymbol{\alpha}^{-1} = \frac{\boldsymbol{\alpha}^*}{N\,(\boldsymbol{\alpha})} \tag{2.8}$$

BIQUATERNIONS (COMPLEX QUATERNIONS)

Everything that can be measured in physics must be real. For this reason, real numbers have found a field of application in every field since the early of science. On the other hand, it is also known that complex numbers are used in mechanical and electrical applications, especially in circuit analysis. Unfortunately this number system only adds 2 dimensions to applications. In 3D applications, vectors are used, but vectors seem to be insufficient in some applications.

Firstly, we can think of real numbers as hyper-complex numbers that are 1-dimensional under any sum, product or any complex number in 2D. We can treat the real numbers as a subset of the complex numbers whose imaginary part is zero. Hamilton discovered the new number system, quaternions, by using complex numbers in 4-dimensional space. In the algebra of quaternions, the combination of two complex numbers is considered, thus a number system is obtained which is expanded in one real and 3 imaginary dimensions. **[7]**

Based on this result, he focused on the triple number system. Quaternion subsumes vector and a real component so we can see that a quaternion has 4-dimension. As in vector space, addition and multiplication operations can be defined in quaternions, but a different method has been developed for division. Quaternions is an algebra that can be expressed in different dimensions, which are complex, divisional and dual. **[8]**

Biquaternions are complex quaternions, so they have 8-dimensions. **[9]** We can extend the applications in Physics such as quantum mechanics, robotics, electromagnetism, etc. by using Biquaternions.

A - Biquaternion Algebra

A biquaternion **Q** is defined as a complex number and $Q_m = a_m + \mathrm{i}\, b_m$ (m = 0, 1, 2, 3) and i^2 = -1, so

$$\mathbf{Q} = Q_0\, \hat{e}_0 + Q_1\, \hat{e}_1 + Q_2\, \hat{e}_2 + Q_3\, \hat{e}_3 \tag{3.1}$$

formed from four different units ($\hat{e}_0$, $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$) by means of the real parameters $\hat{e}_0$ = 1 and $\hat{e}_i$ (i = 1, 2, 3), where $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ are three orthogonal unit elements and $\hat{e}_i^2$ = -1. Here the orthogonal units obey the product of $\hat{e}_i\hat{e}_j = -\,\delta_{ij}\,\hat{e}_0 + \varepsilon_{ijk}\,\hat{e}_k$ rule. With ideas from both vector and matrix algebra, the biquaternion **Q** may be viewed as a linear combination of a scalar Q_0 and a spatial vector $\vec{Q}$ because of

Q = $(a_0 + \mathrm{i}b_0)\,\hat{e}_0 + (a_1 + \mathrm{i}b_1)\,\hat{e}_1 + (a_2 + \mathrm{i}b_2)\,\hat{e}_2 + (a_3 + \mathrm{i}b_3)\,\hat{e}_3 = (a_0 + \mathrm{i}b_0,\ a_1 + \mathrm{i}b_1,\ a_2 + \mathrm{i}b_2,\ a_3 + \mathrm{i}b_3) = (a_0 + \mathrm{i}b_0,\ \vec{a} + \mathrm{i}\vec{b})$. Thus,

$$\mathbf{Q} = Q_0 + \vec{Q} \tag{3.2}$$

If $Q_0 = 0$, **Q** is a purely imaginary number and is called vector biquaternion; when $\vec{Q} = \vec{0}$, **Q** is a real number and is called a scalar biquaternion.

Scalar unit element according to the multiplication in biquaternıon algebra is:

$$1\mathrm{Q} = \mathrm{Q}1 = \mathrm{Q} \tag{3.3}$$

Because of $\hat{e}_0 = 1$.

For two biquaternions such as P and Q, to be equal, their opposite elements must be equal. If P and Q are equal biquaternions, then it must be as:

$$\mathbf{P} = P_0\,\hat{e}_0 + P_1\,\hat{e}_1 + P_2\,\hat{e}_2 + P_3\,\hat{e}_3$$

$$\mathbf{Q} = Q_0\,\hat{e}_0 + Q_1\,\hat{e}_1 + Q_2\,\hat{e}_2 + Q_3\,\hat{e}_3$$

$$P_0 = Q_0,\ P_1 = Q_1,\ P_2 = Q_2,\ P_3 = Q_3 \tag{3.4}$$

The sum or subtraction of two biquaternions is a biquaternion consisting of the sum or subtraction of the opposite elements of these two biquaternions:

$$\mathbf{P} + \mathbf{Q} = (P_0 \pm Q_0)\,\hat{e}_0 + (P_1 \pm Q_1)\,\hat{e}_1 + (P_2 \pm Q_2)\,\hat{e}_2 + (P_3 \pm Q_3)\,\hat{e}_3 \tag{3.5}$$

The multiplication of two biquaternions is:

$$\mathbf{PQ} = (P_0\,\hat{e}_0 + P_1\,\hat{e}_1 + P_2\,\hat{e}_2 + P_3\,\hat{e}_3)(\,Q_0\,\hat{e}_0 + Q_1\,\hat{e}_1 + Q_2\,\hat{e}_2 + Q_3\,\hat{e}_3)$$

$$= P_0\,\hat{e}_0 Q_0\,\hat{e}_0 + P_0\,\hat{e}_0 Q_1\,\hat{e}_1 + P_0\,\hat{e}_0 Q_2\,\hat{e}_2 + P_0\,\hat{e}_0 Q_3\,\hat{e}_3$$

$$+ P_1\,\hat{e}_1 Q_0\,\hat{e}_0 + P_1\,\hat{e}_1 Q_1\,\hat{e}_1 + P_1\,\hat{e}_1 Q_2\,\hat{e}_2 + P_1\,\hat{e}_1 Q_3\,\hat{e}_3$$

$$+ P_2\,\hat{e}_2 Q_0\,\hat{e}_0 + P_2\,\hat{e}_2 Q_1\,\hat{e}_1 + P_2\,\hat{e}_2 Q_2\,\hat{e}_2 + P_2\,\hat{e}_2 Q_3\,\hat{e}_3$$

$$+ P_3\,\hat{e}_3\,Q_0\,\hat{e}_0 + P_3\,\hat{e}_3\,Q_1\,\hat{e}_1 + P_3\,\hat{e}_3\,Q_2\,\hat{e}_2 + P_3\,\hat{e}_3 Q_3\,\hat{e}_3$$

$$= P_0Q_0 + P_0Q_1\hat{e}_1 + P_0Q_2\hat{e}_2 + P_0Q_3\hat{e}_3$$

$$+ P_1Q_0\hat{e}_1 + (-\,P_1Q_1) + P_1Q_2\hat{e}_3 + (-\,P_1Q_3\hat{e}_2)$$

$$+ P_2Q_0\hat{e}_2 + (-\,P_2Q_1\hat{e}_3) + (-\,P_2Q_2) + P_2Q_3\hat{e}_1$$

$$+ P_3Q_0\hat{e}_3 + P_3Q_1\hat{e}_2 + (-\,P_3Q_2\hat{e}_1) + (-\,P_3Q_3)$$

$$= (P_0Q_0 - P_1Q_1 - P_2Q_2 - P_3Q_3)$$

$$+ (P_0Q_1 + P_1Q_0 + P_2Q_3 - P_3Q_2)\,\hat{e}_1$$

$$+ (P_0Q_2 - P_1Q_3 + P_2Q_0 + P_3Q_1)\,\hat{e}_2$$

NaSTech
Journal of Natural Sciences and Technologies
2022, 1(1), 124-134., DOI: 10.5281/zenodo.7323506

$+ (P_0 Q_3 + P_1 Q_2 - P_2 Q_1 + P_3 Q_0)\, \hat{e}_3$

and this shows us that this equation is also equals to:

$$= P_0 Q_0 - \vec{\boldsymbol{P}} \cdot \vec{\boldsymbol{Q}} + P_0 \vec{\boldsymbol{Q}} + \vec{\boldsymbol{P}} Q_0 + \mathrm{i} \vec{\boldsymbol{P}} \times \vec{\boldsymbol{Q}} \qquad (3.6)$$

Because

$\mathbf{PQ} = (P_0\, \hat{e}_0 + P_1\, \hat{e}_1 + P_2\, \hat{e}_2 + P_3\, \hat{e}_3)(\, Q_0\, \hat{e}_0 + Q_1\, \hat{e}_1 + Q_2\, \hat{e}_2 + Q_3\, \hat{e}_3)$

$= (P_0\, \hat{e}_0 Q_0\, \hat{e}_0 + P_0\, \hat{e}_0 Q_1\, \hat{e}_1 + P_0\, \hat{e}_0\, Q_2\, \hat{e}_2 + P_0\, \hat{e}_0 Q_3\, \hat{e}_3)$

$+ (P_1\, \hat{e}_1 Q_0\, \hat{e}_0 + P_1\, \hat{e}_1 Q_1\, \hat{e}_1 + P_1\, \hat{e}_1\, Q_2\, \hat{e}_2 + P_1\, \hat{e}_1 Q_3\, \hat{e}_3)$

$+ (P_2\, \hat{e}_2\, Q_0\, \hat{e}_0 + P_2\, \hat{e}_2\, Q_1\, \hat{e}_1 + P_2\, \hat{e}_2\, Q_2\, \hat{e}_2 + P_2\, \hat{e}_2\, Q_3\, \hat{e}_3)$

$+ (P_3\, \hat{e}_3\, Q_0\, \hat{e}_0 + P_3\, \hat{e}_3\, Q_1\, \hat{e}_1 + P_3\, \hat{e}_3\, Q_2\, \hat{e}_2 + P_3\, \hat{e}_3\, Q_3\, \hat{e}_3)$

$= (P_0 Q_0 + P_0 Q_1\, \hat{e}_1 + P_0 Q_2\, \hat{e}_2 + P_0 Q_3\, \hat{e}_3)$

$+ (P_1 Q_0 \hat{e}_1 - P_1 Q_1 + P_1 Q_2 - P_1 Q_3)$

$+ (P_2 Q_0 \hat{e}_2 - P_2 Q_1 - P_2 Q_2 + P_2 Q_3)$

$+ (P_3 Q_0 \hat{e}_3 + P_3 Q_1 - P_3 Q_2 - P_3 Q_3)$

$= P_0 Q_0 - \vec{\boldsymbol{P}}.\vec{\boldsymbol{Q}} + P_0 \vec{\boldsymbol{Q}} + \vec{\boldsymbol{P}} Q_0 + \mathrm{i}\, \vec{\boldsymbol{P}} \times \vec{\boldsymbol{Q}}$

We can define conjugate of a biquaternion in two ways. The first one is:

$$\bar{Q} = Q_0\, \hat{e}_0 - Q_1\, \hat{e}_1 - Q_2\, \hat{e}_2 - Q_3\, \hat{e}_3 = [Q_0, -\vec{\boldsymbol{Q}}] \qquad (3.7)$$

If we show the biquaternion Q as:

$\mathrm{Q} = (a_0 + \mathrm{i} b_0)\, \hat{e}_0 + (a_1 + \mathrm{i} b_1)\, \hat{e}_1 + (a_2 + \mathrm{i} b_2)\, \hat{e}_2 + (a_3 + \mathrm{i} b_3)\, \hat{e}_3$

so the conjugate of Q is:

$\bar{Q} = (a_0 + \mathrm{i} b_0)\, \hat{e}_0 - (a_1 + \mathrm{i} b_1)\, \hat{e}_1 - (a_2 + \mathrm{i} b_2)\, \hat{e}_2 - (a_3 + \mathrm{i} b_3)\, \hat{e}_3.$

The second conjugation of a biquaternion is getting congugating of all complex coefficient of biquaternion, like:

$$Q^* = Q_0^* \hat{e}_0 - Q_1^*\, \hat{e}_1 - Q_2^*\, \hat{e}_2 - Q_3^*\, \hat{e}_3 \qquad (3.8)$$

So all the parts are:

$Q_0^* = a_0 - \mathrm{i} b_0$

$Q_1^* = a_1 - \mathrm{i} b_1$

$Q_2^* = a_2 - \mathrm{i} b_2$

$Q_3^{*}= a_3$ - ib_3

If P and Q are two biquaternions, their multiplication is:

$(\overline{PQ}) = \overline{QP}$ (3.9)

And their multiplication's conjugation is:

$(PQ)^{*} = Q^{*}P^{*}$ (3.10)

If Q is a biquatenion, its norm is the multiplication of itself and its conjugate, like:

N(Q) = Q$\bar{Q}$ = $\bar{Q}$Q = $Q_0^2 + Q_1^2 + Q_2^2 + Q_3^2$ (3.11)

If the norm of a biquaternion Q doesn't equal to zero, N(Q) ≠ 0, Q can have its inverse, like:

N(Q) = Q$\bar{Q}$ = $\bar{Q}$Q

Q = $\frac{N(Q)}{\bar{Q}}$

So the inverse of Q is:

$Q^{-1} = \frac{\bar{Q}}{N(Q)}$ (3.12)

The multiplication of biquaternion and its intervers equal to one.

QQ^{-1} = Q^{-1}Q = 1 (3.13)

The division of two biquaternions P and Q means multiplication of P and Q^{-1}. However multiplication of two biquaternions has no interchange speciality, so therte are two kinds of division of biquaternions **[10]** :

$$\frac{P}{Q} = \begin{Bmatrix} Q^{-1}P = \frac{\bar{Q}}{N(Q)}\,P \\ PQ^{-1} = P\,\frac{\bar{Q}}{N(Q)} \end{Bmatrix} \quad (3.14)$$

ELECTROMAGNETIC ENERGY CONSERVATION BY BIQUATERNIONS IN HOMOGENOUS MEDIA

In physics, magnetic charges are generally referred to as particles with zero electrical charges and one magnetic pole, so magnetic monopoles can be expressed in accordance with Maxwell's equations for magnetic charges in an isolated medium for quantized electric charges. In homogenous media Maxwell equations are:

$\boldsymbol{\nabla}.\,\mathbf{D} = \rho_e$

$\boldsymbol{\nabla}.\,\mathbf{B} = 0$

$\boldsymbol{\nabla} \times \mathbf{E} = -\,\frac{\partial \vec{B}}{\partial t}$

NaSTech
Journal of Natural Sciences and Technologies
2022, 1(1), 124-134., DOI: 10.5281/zenodo.7323506

$\mathbf{\nabla} \times \mathbf{H} = -\frac{\partial \vec{D}}{\partial t} - \vec{J_e}$

Considering the existence of magnetic monopoles, the symmetrical generalized Maxwell-Dirac differential equations in space with the SI unit system can be expressed in biquaternions as:

$\mathbf{\nabla}.\mathbf{D} = -\frac{\nabla.\mathrm{D}+\overline{\nabla.\mathrm{D}}}{2}$

Here, we accepted c = ħ = 1. Now, we will find $\mathbf{\nabla}.\mathbf{D}$ for biquaternions:

$$\nabla.\mathrm{D} = (\frac{\partial}{\partial x}\hat{e}_1 + \frac{\partial}{\partial x}\hat{e}_2 + \frac{\partial}{\partial x}\hat{e}_3)(D_0\hat{e}_0 + D_1\hat{e}_1 + D_2\hat{e}_2 + D_3\hat{e}_3)$$

$$= \frac{\partial D_0}{\partial x}\hat{e}_1 - \frac{\partial D_1}{\partial x} + \frac{\partial D_2}{\partial x}\hat{e}_3 - \frac{\partial D_3}{\partial x}\hat{e}_2 + \frac{\partial D_0}{\partial y}\hat{e}_2 - \frac{\partial D_1}{\partial y}\hat{e}_3 - \frac{\partial D_2}{\partial y} + \frac{\partial D_3}{\partial y}\hat{e}_1 + \frac{\partial D_0}{\partial z}\hat{e}_3 + \frac{\partial D_1}{\partial z}\hat{e}_2 - \frac{\partial D_2}{\partial z}\hat{e}_1 - \frac{\partial D_3}{\partial z}$$

$$\overline{\nabla.\mathrm{D}} = -\frac{\partial D_0}{\partial x}\hat{e}_1 - \frac{\partial D_1}{\partial x} - \frac{\partial D_2}{\partial x}\hat{e}_3 + \frac{\partial D_3}{\partial x}\hat{e}_2 - \frac{\partial D_0}{\partial y}\hat{e}_2 + \frac{\partial D_1}{\partial y}\hat{e}_3 - \frac{\partial D_2}{\partial y} - \frac{\partial D_3}{\partial y}\hat{e}_1 - \frac{\partial D_0}{\partial z}\hat{e}_3 - \frac{\partial D_1}{\partial z}\hat{e}_2 + \frac{\partial D_2}{\partial z}\hat{e}_1 - \frac{\partial D_3}{\partial z}$$

$$\nabla.\mathrm{D} + \overline{\nabla.\mathrm{D}} = -2\frac{\partial D_1}{\partial x} - 2\frac{\partial D_2}{\partial y} - 2\frac{\partial D_3}{\partial z}$$

$$\mathbf{\nabla}.\mathbf{D} = -\frac{-2\frac{\partial D_1}{\partial x} - 2\frac{\partial D_2}{\partial y} - 2\frac{\partial D_3}{\partial z}}{2} = \frac{\partial D_1}{\partial x} + \frac{\partial D_2}{\partial y} + \frac{\partial D_3}{\partial z} = \rho_e \tag{4.1}$$

And we know

$$\mathbf{\nabla}.\mathbf{B} = 0 \tag{4.2}$$

It is also same for E:

$$\mathbf{\nabla} \times \mathbf{E} = -\frac{\nabla.\mathrm{E}+\overline{\nabla.\mathrm{E}}}{2i} = -\frac{\partial \vec{B}}{\partial t}i \tag{4.3}$$

$$\mathbf{\nabla} \times \mathbf{H} = -\frac{\partial \vec{D}}{\partial t}i - \vec{J_e}i \tag{4.4}$$

Here, ρ_e is electric charge density, $\vec{J_e}$ is current density, **D** electric induction, **E** electric field, **B** is magnetic induction and **H** is magnetic field.

Moreover, electric and magnetic fields are in gauge theory:

$$\mathrm{E} = -\nabla\varphi_e - \frac{\partial \mathrm{A}}{\partial t} - \nabla \times \mathrm{A}' \tag{4.5}$$

$$\mathrm{B} = -\nabla\varphi_m - \frac{1}{\vartheta^2}\frac{\partial \mathrm{A}'}{\partial t} + \nabla \times \mathrm{A} \tag{4.6}$$

Here, φ_e and φ_m are electric and magnetic scaler potentials, A and A′ are electric and magnetic vector potentials and $\vartheta = \frac{1}{\sqrt{\mu\varepsilon}}$ is electromagnetic waves speed. Vector potentials for biquaternions are **[11] [12]** :

$A = (\varphi_e, \vartheta\vec{A}) = \varphi_e\hat{e}_0 + (A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3)\,\vartheta$

$A' = (\vartheta\varphi_m, \vec{A'}) = \vartheta\varphi_m\hat{e}_0 + (A'_1\hat{e}_1 + A'_2\hat{e}_2 + A'_3\hat{e}_3)$

We can show a biquaternion Ψ as:

$$\Psi = E - i\,\vartheta B = (E_1\hat{e}_1 + E_2\hat{e}_2 + E_3\hat{e}_3) - i\,\vartheta(B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3)$$

$$= -\nabla\varphi_e - \frac{\partial A}{\partial t} - \nabla \times A' - i\,\vartheta\,(-\nabla\varphi_m - \frac{1}{\vartheta^2}\frac{\partial A'}{\partial t} + \nabla \times A)$$

POYNTING TEOREM

In this part of this presentation, we took up Poynting Teorem on electrical current form. In Gauss or CGS, Maxwell equations are:

$$\nabla.\mathbf{D} = 4\pi\rho_e \tag{5.1}$$

$$\nabla.\mathbf{B} = 0 \tag{5.2}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{5.3}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} - \frac{4\pi}{c}\vec{J_e} \tag{5.5}$$

If it is accepted that c=1, then these equations become:

$$\nabla.\mathbf{E} = 4\pi\rho_e \tag{5.6}$$

$$\nabla.\mathbf{B} = 0 \tag{5.7}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \vec{B}}{\partial t} \tag{5.8}$$

$$\nabla \times \mathbf{H} = \frac{\partial \vec{D}}{\partial t} - 4\pi\vec{J_e} \tag{5.9}$$

If there is no electric charge density and current density, we can show them as:

$$\nabla.\mathbf{E} = 0 \tag{5.10}$$

$$\nabla.\mathbf{B} = 0 \tag{5.11}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \vec{B}}{\partial t} \tag{5.12}$$

$$\nabla \times \mathbf{B} = \frac{\partial \vec{D}}{\partial t} \tag{5.13}$$

If we multiply (5.10) and (5.12) by B and (5.11) and (5.13) by E:

$$\vec{E}.(\nabla \times \mathbf{B}) + \vec{B}.(\nabla \times \mathbf{E}) + \vec{E}.\frac{\partial \vec{E}}{\partial t} + \vec{B}.\frac{\partial \vec{B}}{\partial t} = 0 \tag{5.14}$$

The directions of the electric and magnetic fields vectors perpendicular to each others are also perpendicular to the direction of the electromagnetic field waves. The direction of electromagnetic waves has also the direction of energy flux $\vec{S}$'s direction.

$$\vec{S} = \vec{E} \times \vec{B} \tag{5.15}$$

If energy flux $\vec{S}$ is also poynting vector and u is electromagnetic energy density:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{E} + \vec{B}) = \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}.\vec{E} + \vec{B}.\vec{B}) = \mathbf{\nabla}.\mathbf{S} \tag{5.16}$$

We can show $\mathbf{\nabla}.(\vec{A} \times \vec{B}) = \vec{B}.(\mathbf{\nabla} \times \mathbf{A}) + \vec{A}.(\mathbf{\nabla} \times \mathbf{B})$ for two biquaternions so (5.15) and (5.16) are rearranged:

$$\mathbf{\nabla}.\mathbf{S} = \mathbf{\nabla}.(\vec{E} \times \vec{B}) = \vec{B}.(\mathbf{\nabla} \times \mathbf{E}) + \vec{E}.(\mathbf{\nabla} \times \mathbf{B}) = 0$$

$$\frac{\partial u}{\partial t} = \mathbf{\nabla}.\mathbf{S} = 0 \tag{5.17}$$

This equation is named as "energy conservation equation" and also "Poynting Theorem". **[13]**

CONCLUSION

In this study the Poynting Theorem and Electromagnetic Conservation are tried to be represented with biquaternion algebra. It was shown that the biquaternionic expressions are nearly same with vector algebra. Biquaternion algebra used in this study have extended to complex number system. Biquaternions are useful and effective mathematical structure for energy conservation.

REFERENCES

1- J. C. K. Chou, "*Quaternion Kinematic and Dynamic Differantial Equations*", IEEE Transactions on Robotics and Automation, Vol. 8, No: 1, February 1992

2- I. K. Hong and C. S. Kim, "*Quaternion Electromagnetism and Relation with 2- Spinor Formalism",* arXiv:1902.09773v2, June 2019

3- J. Funda, P.P. Richard, IEEE Trans. Robot. Autom. **6**, 348–356 (1990)

4- K. Gürlebeck, W. Sprössig, "*Quaternionic and Clifford Calculus for Physicists and Engineers",* Wiley, Chichester, 1997

5- https://en.wikipedia.org/wiki/Poynting%27s_theorem

6- J. C. K. Chou, "*Quaternion Kinematic and Dynamic Differantial Equations*", IEEE Transactions on Robotics and Automation, Vol. 8, No: 1, February 1992

7- V. Majernik, Adv. Appl. Clifford Alg. **9**, 119 (1999)

8- T. Y. Lam, D. B. Leep, J. Tıgrol, "*Biquaternion algebra and quaretic extensions*", Publication Mathematiquaes De L'I.H.E.S, volume 77, 1993

9- P. R. Girard, P. Clarysse, R. Pujol, R. Goutte, P. Delachartre, "*Hyperquaternions: A new Tool for Physics",* Advances in Applied Clifford Algebeas, June 2018

10- W. R. Hamilton, "*Elements of Quaternions Vol I, II, III*", Chelsea, New York, USA, 1899

11- B. Jancewicz, "*Multivectors and Clifford Algebra in Electrodynamics",* World Scientific, Singapore, 1989

12- K.V. Khmelnytskaya, V. V. Kravchenko, “*Biquaternions for analytic and numerical solution of equations of electrodynamics*”, https://arxiv.org/abs/0902.3490

13- A. Waser, “*Applications of Bi-Quaternions in Physics”,* Researchgate, July 2000